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Standard Definitions and Methods of Measurement for

PIEZOELECTRIC VIBRATORS

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CONTENTS

DEFINITIONS AND METHODS OF MEASUREMENT FOR PIEZOELECTRIC VIBRATORS

INTRODUCTION:

This Standard is a revision of the IRE Standard on Piezoelectric Crystals—The Piezoelectric Vibrator: Definitions and Methods of Measurement, 1957 (57 IRE 14. S1)[1] and a continuation of Standards in this field.[2, 3, 4]

An introductory review of the equivalent electric circuit of a piezoelectric vibrator and its parameters is followed by a discussion of the determination of these parameters by the transmission method. This method was published in 1951[5] and became the basis for the 1957 IRE Standard.[1] Since that time, a thorough investigation of the transmission method has resulted in more precise expressions which permit a more accurate evaluation of the parameters.[6] This method is suitable for frequencies up to about 30 MHz for the commonly encountered ranges of the capacitance ratio r and the figure of merit M, provided that errors due to instrumentation are taken into account. The equations presented in this Standard have been formulated to correct these errors.

1. CLASSIFICATION OF PHENOMENA

1.1 *The Piezoelectric Vibrator and its Equivalent Electric Circuit*

A piezoelectric vibrator consists of an element usually in the form of a plate, bar or ring cut from a piezoelectric material and has electrodes attached to or supported near the element to excite one of its resonance frequencies.

The electrical behavior of a lightly damped mechanical vibrating system which is excited piezoelectrically through electrodes forming a two-terminal network, can be represented in the vicinity of any mechanical resonance by an equivalent electric circuit (Figure 1) which consists of a capacitance C_1, inductance L_1, and resistance R_1 in series, shunted by the parallel capacitance C_0. The parameters are independent of frequency for isolated modes of motion. Generally, the mode in question is sufficiently isolated from other modes to permit this assumption. When this is not true, the equations and measuring methods outlined herein do not apply. For identification of symbols used in this Standard, see Table 1A.

1.2 *Parameters of Piezoelectric Vibrators*

The fundamental parameters C_1, L_1, R_1, and C_0 define the equivalent electric circuit shown in Figure 1 and all other parameters may be derived from them. At a given frequency the parameters of the equivalent electric circuit generally approach constant values as the amplitude of vibration approaches zero. The amplitude which can be tolerated before the parameters are appreciably affected varies widely between vibrators of various types and can only be determined by experiment.

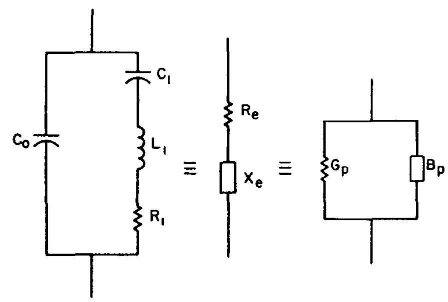

FIGURE 1

Equivalent Electric Circuit of a Piezoelectric Vibrator Near a Resonance.

The equation for the impedance Z or admittance Y

$$Z = \frac{1}{Y} = \frac{j}{\omega C_0} \, \frac{\Omega - j\delta}{1 - \Omega + j\delta} \qquad (1)$$

of the equivalent electric circuit of the piezoelectric vibrator is the basic equation describing the relationships between the various parameters. In equation (1)

$$\Omega = \frac{f^2 - f_s^2}{f_p^2 - f_s^2} \quad \text{and} \quad \delta = 2\pi f C_0 R_1$$

are the normalized frequency factor and the normalized damping factor, respectively. See Table 1A for definitions of f_p, f_s, and the other symbols used in equation (1) and for other essential parameters. The characteristic frequencies of equation (1) are defined in Table 2.

The magnitude of the impedance of the equivalent electric network ($|Z|$), its resistive component (R_e), its reactive component (X_e), and the reactance X_1 of the L_1, C_1, R_1 branch are plotted as functions of frequency in Figure 2, for the purpose of defining the different characteristic frequencies. Z_m and Z_n denote minimum and maximum impedance respectively, and R_r, R_a the impedances at zero phase angle. These curves, however, have only qualitative character and do not represent a particular piezoelectric vibrator.[7]

For further clarification, the impedance and admittance circles of a piezoelectric vibrator are reproduced in Figure 3. However, the circle representation of the impedance or admittance of a piezoelectric vibrator is valid only if the circle diameter of the admittance diagram is large compared with the change of $2\pi f C_0$ in the resonance range or if $r \ll Q^2$, which is fulfilled in most vibrators. If the latter conditions are not fulfilled, the admittance curve shows a cissoidal character. Throughout the remainder of this Standard, it is assumed that the impedance (or admittance) of the vibrator can be represented by a circle diagram. Table 3 gives data for Q, r, and Q^2/r for various types of vibrators, indicating that this assumption is valid for all practical cases.

It is necessary to make approximations in deriving practical equations for general use. It is the error of these approximations, in addition to the errors of instrumentation, that govern the overall accuracy of the experimentally derived parameters.

As a first approximation sufficient for many practical purposes, the following assumptions can be made: $f_m \approx f_r \approx f_s$ and $f_a \approx f_n \approx f_p$.

More exact relations between the characteristic frequencies f_m, f_r, f_a, f_p, f_n, and the series resonance frequency f_s of a vibrator, valid for the figure of merit $M > 10$ and the capacitance ratio $r > 10$, are shown in Table 4. These relationships have been derived by various authors[8, 9] under the assumption that $M \gg 1$.

The separation between parallel and series resonance frequencies is given by:

$$\frac{f_p^2 - f_s^2}{f_s^2} = \frac{C_1}{C_0} = \frac{1}{r} \qquad (2)$$

The approximation

$$\frac{f_p - f_s}{f_s} = \sqrt{1 + r^{-1}} - 1$$
$$= \frac{1}{2r}\left(1 - \frac{1}{4r} + \ldots\right) \approx \frac{1}{2r}$$
$$= \frac{1}{2}\frac{C_1}{C_0} \qquad (3)$$

can be used for larger values of r (for example, when r is greater than 25 the error is less than 1 percent.)

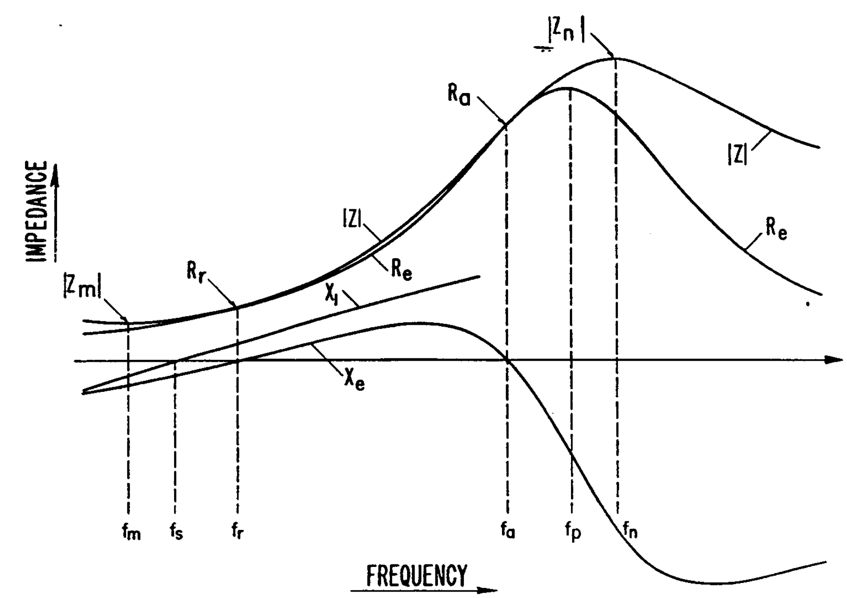

FIGURE 2

Impedance $|Z|$, Resistance R_e, Reactance X_e, and Series Arm Reactance X_1 of a Piezoelectric Vibrator as a Function of Frequency. Z_m *and* Z_n denote minimum and maximum impedance, R_r *and* R_a the impedances at zero phase angle. For the meaning of the different frequencies, see Table 1A and Figure 2.

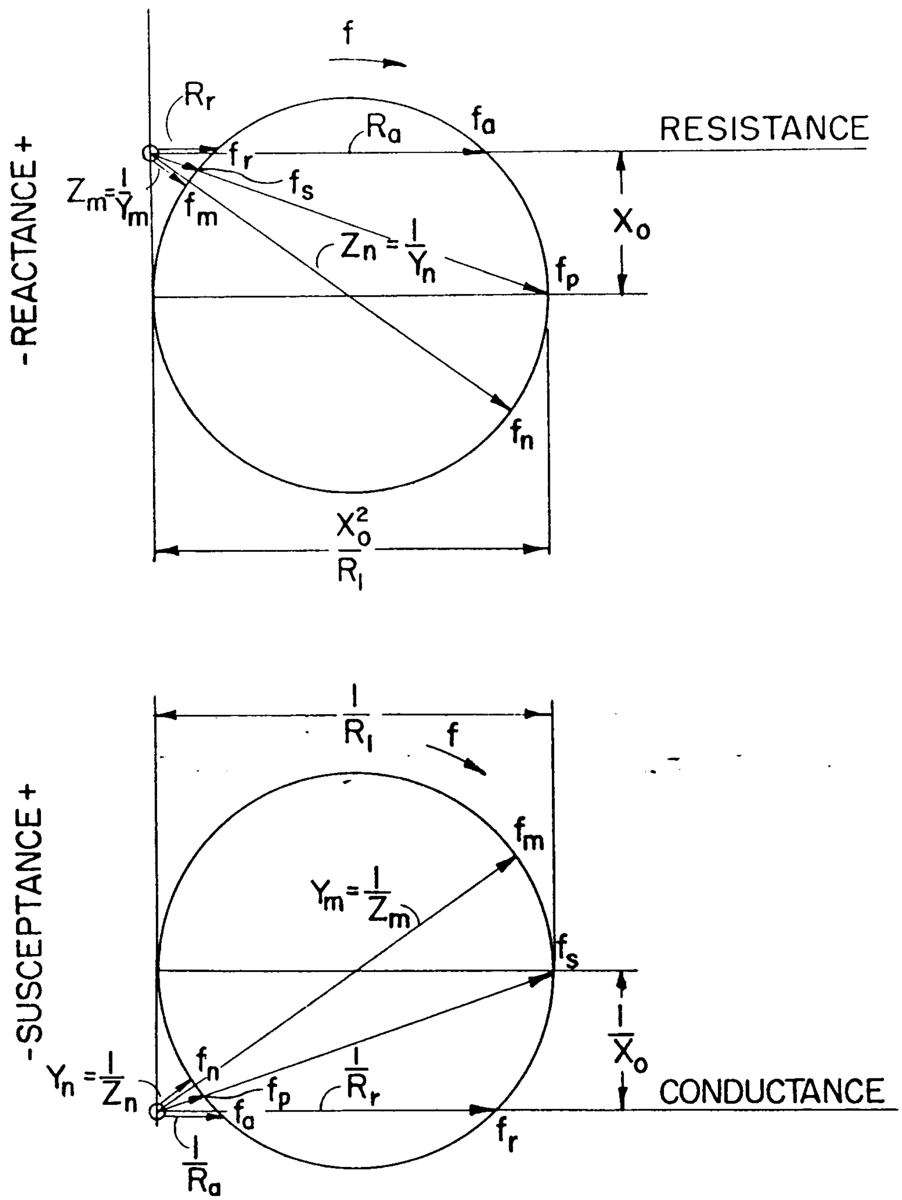

FIGURE 3

Impedance and Admittance Diagram of a Piezoelectric Vibrator. The symbols conform with those in Table 1A and Figure 2.

2. TRANSMISSION CIRCUIT METHOD OF MEASURING THE PARAMETERS OF THE EQUIVALENT ELECTRIC CIRCUIT

2.1 *Measurement, General*

This method is based on measuring the frequency and impedance at maximum transmission (maximum transfer impedance) of a π-network containing the equivalent electric circuit of the vibrator under test in the series branch, as shown by Figure 4. The frequency f_{mT} at maximum transmission (maximum output voltage) is measured both with and without the capacitance C_L in series with the vibrator. From these measurements, the motional resonance frequency f_s and the motional capacitance C_1 of the vibrator can be determined. The value for R_1 can be obtained by substitution of a resistance R_{st} in place of the vibrator to obtain the same output voltage.

All symbols describing the transmission method are given in Tables 1A and 1B.

Table 5 (taken from [6]) contains, in the second column, a compilation of the expressions for R_1, and, in the third column, the exact deviation of the frequency at minimum transfer admittance f_{mT} from the motional resonance frequency f_s of the vibrator. The exact solutions shown in Table 5 can be simplified if the assumptions mentioned in the left-hand column of the table are made. When the parallel inductance L_0 is not used, then $b = 1$.

2.2 *Transmission Measurement Circuit*

Figure 4 shows a schematic of the transmission circuit. The measuring circuit consists of a constant current source in the form of a variable frequency oscillator, the transmission network, and a voltmeter. The piezoelectric vibrator is represented by its equivalent electric circuit. The network is symmetrical with respect to the input and output. The capacitances C_T, which shunt the terminating resistances R_T, represent stray elements which affect the accuracy of measurements as shown in Table 5. The oscillator must have a high degree of purity of output waveform to an extent consistent with the requirements of the individual vibrator under test.*

The inductance L_0, connected across the vibrator, serves to resonate the shunt capacitance C_0 of the vibrator at f_s. This added component improves the accuracy of measurement as explained in Section 2.3.1.

An important source of stray capacitance to ground that must be considered occurs at the junction of the crystal unit and the capacitor C_L (Figure 4). This stray capacitance is composed of two parts; that associated with the crystal unit under test and that associated with C_L. When the magnitude of these stray reactances is large compared with the magnitude of the termination impedance, the distributed capacitance to ground of the crystal unit and the distributed capacitance to ground of C_L at the junction may be treated in a first approximation as being in parallel with C_L. When measuring the parameters of crystal units by other than the transmission method, consideration must be given to the stray capacitance of each terminal of the crystal unit to ground. This is of course an important consideration in the use of crystal units in network and frequency control applications.

The voltmeter which is placed at the output, measures the voltage e_2 as the frequency of the input is varied.

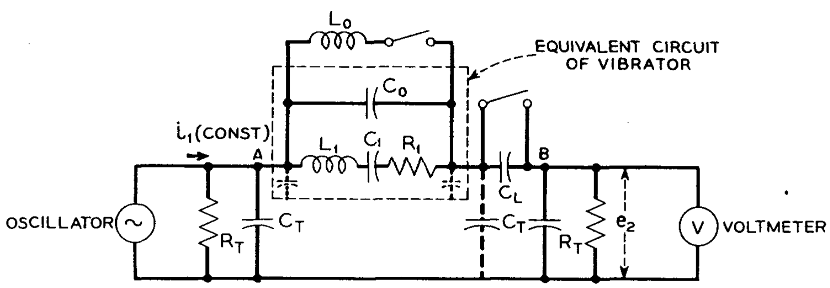

FIGURE 4

Schematic of Transmission Circuit Method.

2.3 *Procedure for Measurement and Determination of the Parameters*

2.3.1 *Motional Resistance R_1*

The motional resistance is measured by adjusting the frequency of the oscillator to obtain maximum transmission (maximum e_2), of the vibrator. The resistor is adjusted to the particular value R_{st} for which the maximum transmission value e_2 is equal to the value obtained with the vibrator. The general relation between R_1 and R_{st} is given in equation (4), Table 5. Assuming that $C_T = 0$, R_1 can and then substituting a resistor R_{st} in place be calculated from equation (4b), Table 5. When the compensating inductance L_0 is not used, the error in R_1 will be

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* For most practical purposes the following requirements are adequate: Harmonics greater than 30 dB below the main signal, frequency stability better than 1 part per 10^6 and amplitude change less than 10 percent during the measurement period.

$$100 \cdot \left(\frac{R_{st}}{X_o} \right)^2 \left(\frac{4R_T}{R_{st}} + 1 \right) \text{ percent.} \quad (5)$$

Proper adjustment of L_o for $b = O$ (Table 1B, first line) reduces this error to zero (Table 5, equation 4f).

2.3.2 *Motional Capacitance C_1 and Inductance L_1*

The motional capacitance C_1 is determined by measuring the frequency of maximum transmission f_{mT} using one or more load capacitances* C_L connected successively in series with the vibrator (see Figure 4). When $M \gg 1$, which is usually the case, the readings obtained by use of two different load capacitors C_{L1} and C_{L2} can be combined so that

$$C_1 = \frac{2\,\Delta C_L}{f_s} \quad \frac{\Delta f_1\,\Delta f_2}{\Delta f} \quad (6)$$

where

$$\begin{aligned} \Delta C_L &= C_{L2} - C_{L1} \\ \Delta f &= f_{sL1} - f_{sL2} \\ \Delta f_1 &= f_{sL1} - f_s \\ \Delta f_2 &= f_{sL2} - f_s \end{aligned} \quad (7)$$

and f_{sL1} and f_{sL2} are the motional resonance frequencies of the vibrator in series with C_{L1} and the vibrator in series with C_{L2} respectively. The frequencies of maximum transmission may be used in equation (7) instead of the respective motional resonance frequencies resulting in an error in C_1 of less than 3 percent for $(Q^2/r) \geqq 80$. To obtain higher accuracy, the motional resonance frequencies have to be calculated from the frequencies of maximum transmission according to Section 2.3.3 and Table 5. As the equations in Table 5 are based on the equivalent electric circuit shown in Figure 1, equations (8) have to be used to obtain the corresponding parameters for the combination of vibrator plus C_L. When a capacitance C_L is in series with an electroded element, the diagram shown in Figure 5 applies.[10] The two circuits are equivalent having the following relationships:

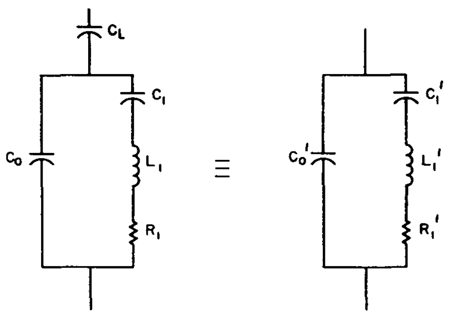

FIGURE 5
Equivalent Electric Circuit of a Piezoelectric Vibrator with a Series (Load) Capacitance C_L.

$$\begin{aligned} L'_1 &= L_1 \left(1 + \frac{C_o}{C_L} \right)^2 \\ C'_1 &= C_1 \frac{1}{\left(1 + \frac{C_o}{C_L}\right)^2 \left(1 + \frac{C_1}{C_o + C_L}\right)} \\ R'_1 &= R_1 \left(1 + \frac{C_o}{C_L} \right)^2 \\ C'_o &= \frac{C_o \quad C_L}{C_o + C_L}. \end{aligned} \quad (8)$$

When the frequencies of maximum transmission are measured using different values for C_L, it is important to maintain the current through the vibrator constant to within 10 percent as indicated by the voltmeter.

The inductance L_1 follows from

$$L_1 = \left(\omega_s^2\, C_1 \right)^{-1} \quad (9)$$

when the values for ω_s and C_1 are known.

2.3.3 *Motional Resonance Frequency f_s*

The frequency of the oscillator is adjusted for maximum transmission with the piezoelectric vibrator inserted in the network shown in Figure 4. This is the frequency of maximum transmission f_{mT}. At the first approximation, f_{mT} is equal to the frequency of minimum impedance f_m and the motional resonance frequency f_s of the vibrator. If higher accuracy for f_s is required, Table 5 should be consulted which gives the relationship between f_{mT} and f_s as the ratio

$$\frac{f^2_{mT}}{f^2_s} - 1$$

for different degrees of approximation as a function of the network parameters. When the shunting inductance L_o is omitted, equation (10b) in Table 5 becomes

$$\frac{f^2_{mT}}{f^2_s} - 1 = \frac{1}{r} \left\{ 1 + \frac{2}{M^2\left(1 + \frac{4R_T}{M^2 R_1}\right) \left[1 - \sqrt{1 + \frac{4}{M^2\left(1 + \frac{4R_T}{M^2 R_1}\right)^2}} \right]} \right\}. \quad (11a)$$

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* See Section 2.2 for the effect of stray capacitance on C_L.

When $M^2 \gg 1$ is fulfilled for the resonator, the general formula (10) reduces to

$$\frac{f^2_{mT}}{f^2_s} - 1 \approx \frac{-1}{M^2 r}\left(\frac{4R_T}{R_1} + 1\right) \quad (11b)$$

and since the right-hand side of equation (11b) is usually much smaller than unity, it further reduces to

$$\frac{f_{mT} - f_s}{f_s} = \frac{\Delta f}{f_s} \approx \frac{-1}{2M^2 r}\left(\frac{4R_T}{R_1} + 1\right). \quad (11c)$$

In most instances, $M^2 \gg 1$, and the approximate equation (11c) is satisfactory. When this condition is not fulfilled, the exact formula (11a) must be used.

2.3.4 *Network Requirements*

The accuracy of the results increases as the following conditions are fulfilled:

(1) Stray capacitance C_{A-B} between terminals A and B low compared to vibrator capacitance C_0, ($C_0 \gg C_{A-B}$).

(2) Reactance of stray capacitance C_{A-B} high compared to series resistance R_1, ($|X_{A-B}| \gg R_1$).

(3) The reactance of leads connecting vibrator is low compared to reactance of C_0.

In the case of vibrators with low figure of merit M, it is advisable to use a shunting coil L_0 connected in parallel with the vibrator. If the combination $L_0\ C_0$ is tuned to the motional resonance frequency f_s of the vibrator, $b = O$ and the measurement is simplified. Table 5, equations (4a) and (10a), supplies resistance and frequency values for this condition. It is seen that when $b = O$ and $M_T \gg 1$ (condition 1 above), then $f_{mT} = f_s$ and $R_{st} = R_1$.

2.3.5 *Shunt Capacitance C_0*

The shunt capacitance C_0 of the equivalent electric circuit of a vibrator is slightly smaller than the measured value for a free piezoelectric element and slightly greater than the measured value for a piezoelectric element in clamped condition. The exact value of the dielectric permittivity depends upon the mode of vibration. This has to be considered when greater accuracy is required. Details are discussed in the IEEE Standard on Piezoelectric Crystals: Determination of the Elastic, Piezoelectric, and Dielectric Constants—The Electromechanical Coupling Factor, 1958.[3]

There is no direct method for measuring C_0 precisely. However, in nearly all practical cases it is adequate to regard as C_0 the mean value of the shunt capacitances C_{01} and C_{02} of the resonator obtained at two frequencies, equidistant above and below the resonance frequency and sufficiently removed from the latter for the impedance to be independent of any response. C_{01} and C_{02} can be measured by means of an impedance bridge or a Q meter.

It should be noted that C_0 is the shunt capacitance between the two electrodes of the resonator. As pointed out in Section 2.2, the capacitances of both of the electrodes to ground are important elements in many network and frequency control applications. Proper use of the transmission circuit method for the measurement of C_1 requires knowledge of at least the capacitance to ground of that electrode of the resonator which is connected to C_L (see Figure 4).

Therefore, in the general case, it is necessary to consider the crystal unit as a three terminal network and to evaluate C_0 and the stray capacitances of the two electrodes to ground from open and short-circuit measurements according to the techniques customarily employed when dealing with two-port devices.

The crystal enclosure remains at ground potential during the entire series of measurements required for evaluation of the resonator parameters. For this purpose it may be found desirable to provide glass-enclosed crystal units with metal shells.

2.4 *Numerical Examples*

The IEEE Standard on Piezoelectric Crystals, 1957,[1] shows the mean deviation of f_s from its true value, due to detector sensitivity S alone, as

$$x \approx \frac{\Delta f}{f_s} = \frac{1}{\sqrt{2}}\left(\frac{2R_T}{R_1} + 1\right)\frac{\sqrt{S}}{Q}. \quad (12)$$

This equation is valid if $4/M_T^2 \ll 1$ and $b^2/M^2 \ll 1$. The magnitude of the mean deviation is plotted in Figure 6 for various values of S as a function of Q.

The equations in Table 5 give the corrections necessary to obtain R_1 from R_{st} and f_s from f_{mT} respectively in terms of the vibrator and network parameters. When the assumptions leading to the simplified relations (5) and (11c) are met, the magnitude of these corrections can be obtained from the graphs shown in Figures 7 and 8 (see [6]).

The solid lines in these graphs further assume that $2\ (R_T/R_1) \ll 1$ (Figure 6), $4\ (R_T/R_{st}) \ll 1$ (Figure 7), and $4\ (R_T/R_1) \ll 1$ (Figure 8). The ordinate values of the graphs (Figures 6 to 8) can be modified easily when these relationships are not fulfilled. The examples shown in the graphs refer to the vibrators listed in Figure 6 and illustrate that R_1 generally differs from R_{st} by less than 2 percent (Figure 7) and that, excluding ceramics, the differences between f_s and f_{mT} are of the order of $1 \cdot 10^{-6}$ (Figure 8). Shunt coils were not used for these measurements.

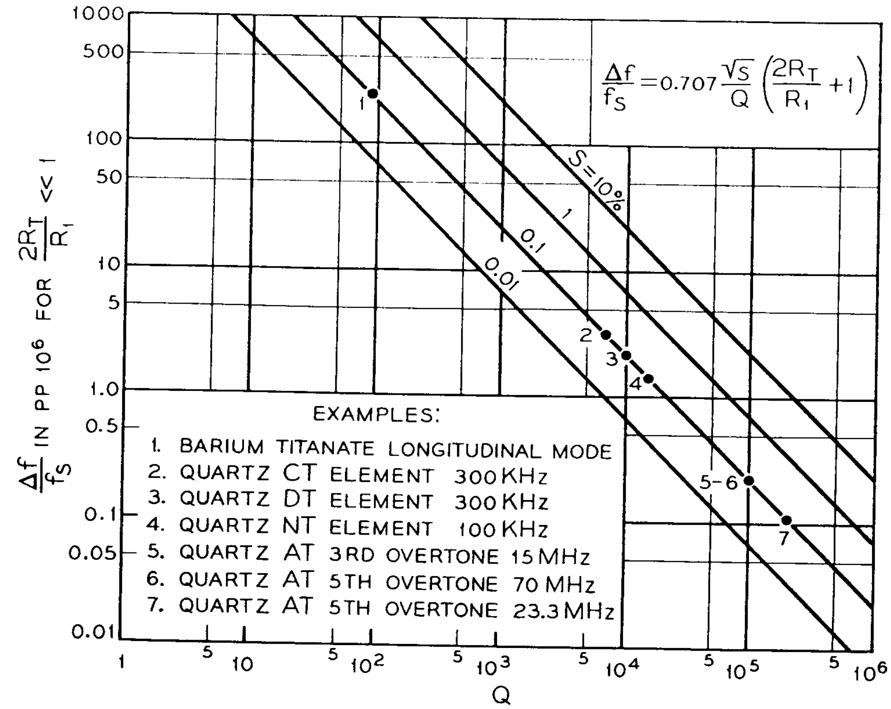

FIGURE 6

Mean Deviation $\Delta f/f_s$ Due to Voltmeter Sensitivity S.

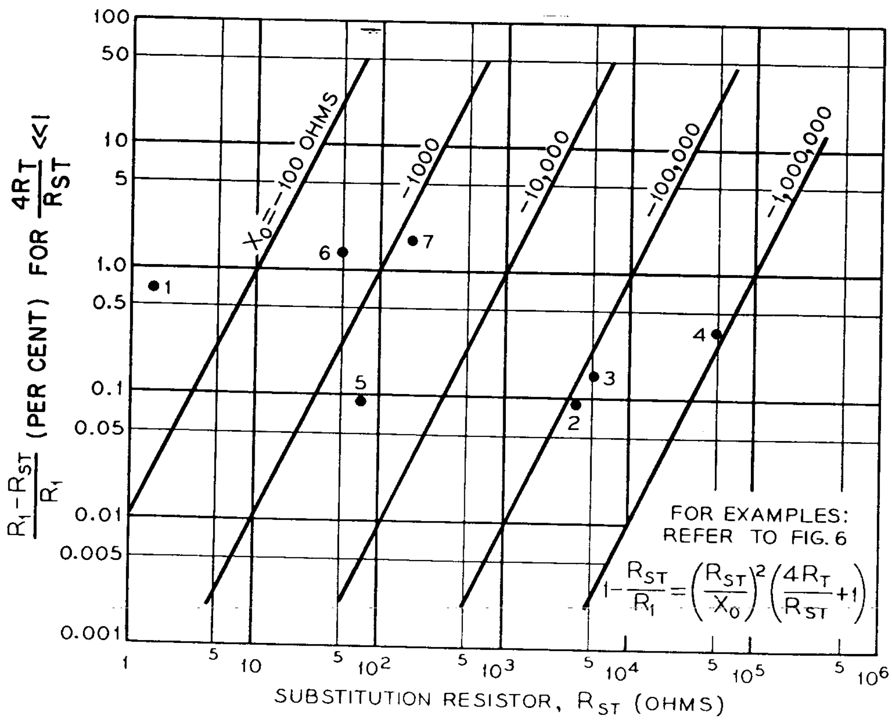

FIGURE 7

Resistance Mean Deviation.

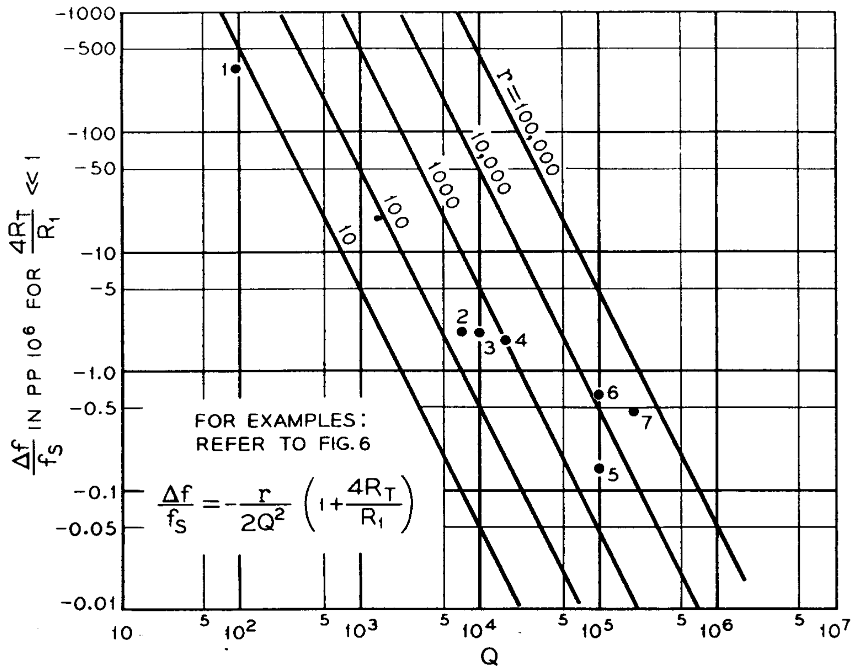

FIGURE 8

Mean Deviation $\Delta f/f_S$ Due to the Vibrator and Circuit Parameters.

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TABLE 1A

LIST OF SYMBOLS USED FOR THE EQUIVALENT ELECTRIC CIRCUIT OF A PIEZOELECTRIC VIBRATOR

Symbols	Meaning	SI Units	References		
			Equations	Tables	Figures
B_p	Equivalent parallel susceptance of vibrator	mho		2	
C_o	Shunt (parallel) capacitance in the equivalent electric circuit	farad	2, 3, 4, 8	5	1, 4
C_1	Motional capacitance in the equivalent electric circuit	farad	2, 3, 4, 6, 8, 9	5	1, 4
f	Frequency	hertz			3
f_a	Antiresonance frequency, zero susceptance	hertz		2, 4	2, 3
f_m	Frequency of maximum admittance (minimum impedance)	hertz		2, 4	2, 3
f_n	Frequency of minimum admittance (maximum impedance)	hertz		2, 4	2, 3
f_p	Parallel resonance frequency (lossless) $= \dfrac{1}{2\pi\sqrt{L_1 \dfrac{C_1 C_o}{C_1 + C_o}}}$	hertz	2, 3	2, 4	2
f_r	Resonance frequency, zero susceptance	hertz		2, 4	2, 3
f_s	Motional (series) resonance frequency $= \dfrac{1}{2\pi\sqrt{L_1 C_1}}$	hertz	2, 3, 6, 7, 9 11a, 11b, 11c, 12	2, 4	2, 3, 6, 8
G_p	Equivalent parallel conductance of vibrator		1		
L_1	Motional inductance in the equivalent electric circuit	henry	8, 9		1, 4, 5
M	Figure of merit of a vibrator $= \dfrac{Q}{r}$ $M = \dfrac{1}{\omega_s C_o R_1}$	dimensionless	10, 11a, 11b	3, 4, 5	
Q	Quality factor $Q = \dfrac{\omega_s L_1}{R_1} = \dfrac{1}{\omega_s C_1 R_1} = rM$	dimensionless	12	3	6, 8
r	Capacitance ratio $r = \dfrac{C_o}{C_1}$	dimensionless	2, 3, 10, 11	2, 3, 4, 5	8
R_a	Impedance at zero phase angle near antiresonance	ohm			2, 3

TABLE 1A (Continued)

Symbols	Meaning	SI Units	References: Equations	Tables	Figures
R_e	Equivalent series resistance of vibrator	ohm			1, 2
R_r	Impedance at f_r zero phase angle	ohm			2, 3
R_1	Motional resistance in the equivalent electric circuit	ohm	4, 8, 10 11a, 11b, 11c, 12	2, 5	1, 3, 4, 6 7, 8
X_e	Equivalent series reactance of vibrator	ohm			1, 2
X_o	Reactance of shunt (parallel) capacitance at series resonance $X_o = \frac{1}{\omega_s C_o}$	ohm	1, 4, 5	5	3, 7
X_1	Reactance of motional (series) arm of vibrator $X_1 = \omega L_1 - \frac{1}{\omega C_1}$	ohm		2	2
Y	Admittance of vibrator $Y = G_p + jB_p = \frac{1}{Z}$	mho	1		
Y_m	Maximum admittance of vibrator	mho			3
Y_n	Minimum admittance of vibrator	mho			3
Z	Impedance of vibrator $Z = R_e + jX_e$	ohm	1		
Z_m	Minimum impedance of vibrator	ohm			3
Z_n	Maximum impedance of vibrator	ohm			3
$\|Z\|$	Absolute value of impedance of vibrator $Z = \sqrt{R_e^2 + X_e^2}$	ohm		2	2
$\|Z_m\|$	Absolute value of impedance at f_m (minimum impedance)	ohm			2
$\|Z_n\|$	Absolute value of impedance at f_n (maximum impedance)	ohm			2
δ	Normalized damping factor $\delta = \omega C_o R_1$	dimensionless	1	2	
Ω	Normalized frequency factor $\Omega = \frac{f^2 - f_s^2}{f_p^2 - f_s^2}$	dimensionless	1	2	
ω	Circular (angular) frequency $\omega = 2\pi f$	hertz		2	
ω_s	Circular frequency at motional resonance $\omega_s = 2\pi f_s$	hertz			

TABLE 1B

LIST OF SYMBOLS USED FOR THE TRANSMISSION NETWORK

Symbols	Meaning	SI Units	References		
			Equations	Tables	Figures
b	Normalized compensation factor $1 - \frac{1}{4\pi^2 f_s^2 C_0 L_0}$	dimensionless	4, 10	5	
B	Normalized admittance factor	dimensionless	10	5	
C	Normalized admittance factor	dimensionless	10	5	
C_{A-B}	Stray capacitance between the terminals **A—B** (Figure 4)	farad			
C_L	Load capacitance	farad	6		4
C_T	Shunt capacitance terminating transmission circuit	farad	4, 10	5	4
C_{L1}	Load capacitance	farad	7		
C_{L2}	Load capacitance	farad	7		
e_2	Output voltage of transmission network	volt			4
f_{mT}	Frequency of maximum transmission	hertz	10		
f_{sL1}	Motional resonance frequency of combination of vibrator and C_{L1}	hertz	7		
f_{sL2}	Motional resonance frequency of combination of vibrator and C_{L2}	hertz	7		
i_1	Input current to transmission network	ampere			4
L_0	Compensation inductance shunting vibrator	henry			4
M_T	Figure of merit of transmission network termination $M_T = \frac{1}{2\pi f_s C_T R_T} = \frac{X_T}{R_T}$	dimensionless	4, 10	5	
R_T	Shunt resistance termination of transmission network	ohm	4, 11a, 11b, 11c, 12	5	4, 6, 7, 8
R_{st}	Standard resistor	ohm	4, 5	5	7

TABLE 1B (Continued)

Symbols	Meaning	SI Units	References		
			Equations	Tables	Figures
S	Detector sensitivity smallest detectable current change/current	dimensionless	12		6
x	Normalized frequency factor $x = \frac{f^2}{f_s^2} - 1 = \frac{1\Omega}{r}$	dimensionless	12		
X_{A-B}	Reactance of stray capacitance C_{A-B}	ohm			
X_T	Reactance of C_T at the motional resonance frequency $X_T = \frac{1}{2\pi f_s C_T}$	ohm	4	5	
x_{mT}	Normalized frequency factor at the frequency of maximum transmission	dimensionless		5	
ΔC_L	$\Delta C_L = C_{L2} - C_{L1}$	farad	6, 7		
Δf	$\Delta f = f_{sL1} - f_{sL2}$	hertz	6, 7		6, 8
Δf_1	$\Delta f_1 = f_{sL1} - f_s$	hertz	6, 7		
Δf_2	$\Delta f_2 = f_{sL2} - f_s$	hertz	6, 7		

TABLE 2

SOLUTIONS FOR THE VARIOUS CHARACTERISTIC FREQUENCIES

Characteristic Frequencies	Meaning	Condition	Constituent Equation for Frequency	Root	57 IEEE 14.S1[1]
f_m	Frequency of maximum admittance (minimum impedance)	$\frac{d\lvert Z \rvert}{d\omega} = 0$	$(\Omega^2 + \delta^2)^2 - 2\delta^2(\Omega + r) - 2\Omega r(1 - \Omega) - \Omega^2 = 0$	lower*	f_m
f_s	Motional (series) resonance frequency	$X_1 = 0$	$\Omega = 0$		f_s
f_r	Resonance frequency	$X_e = B_p = 0$	$\Omega(1 - \Omega) - \delta^2 = 0$	lower	f_r
f_a	Antiresonance frequency	$X_e = B_p = 0$	$\Omega(1 - \Omega) - \delta^2 = 0$	upper	f_a
f_p	Parallel resonance frequency (lossless)	$X_e \big\rvert_{R_1 = 0} = \infty$	$\Omega = 1$		f_p
f_n	Frequency of minimum admittance (maximum impedance)	$\frac{d\lvert Z \rvert}{d\omega} = 0$	$(\Omega^2 + \delta^2)^2 - 2\delta^2(\Omega + r) - 2\Omega r(1 - \Omega) - \Omega^2 = 0$	upper*	f_n

* Refers to real roots; complex roots to be disregarded.

TABLE 3

Type of Piezoelectric Vibrator	$Q = Mr$	r	Q^2/r min
Piezoelectric Ceramics	90 — 500	2 — 40	200
Water-Soluble Piezoelectric Crystals	200 — 50,000	3 — 500	80
Quartz	10^4 — 10^7	100 — 50,000	2000

Minimum Values for the Ratio Q^2/r to be Expected for Various Types of Piezoelectric Vibrators

TABLE 4

Characteristic Frequency	1st Approximation		2nd Approximation	
	$\frac{f}{f_s}$	Deviation $\frac{\Delta f}{f_s}$ From More Precise Value	$\frac{f}{f_s}$	Deviation $\frac{\Delta f}{f_s}$ From More Precise Value
f_m	$\frac{f_m}{f_s} = 1$	$-\frac{1}{2M^2 r}$	$\frac{f_m}{f_s} = \sqrt{1 + \frac{1}{2r}\left[1 - \sqrt{1 + \frac{4}{M^2}}\right]}$	$\frac{1}{2M^4 r^2}$
f_r	$\frac{f_r}{f_s} = 1$	$\frac{1}{2M^2 r}$	$\frac{f_r}{f_s} = \sqrt{1 + \frac{1}{2r}\left[1 - \sqrt{1 - \frac{4}{M^2}}\right]}$	$\frac{1}{2M^4 r^2}$
f_a	$\frac{f_a}{f_s} = 1 + \frac{1}{2r}$	$-\frac{1}{2M^2 r}\left(\frac{1}{r} + 1\right)$	$\frac{f_a}{f_s} = \sqrt{1 + \frac{1}{2r}\left[1 + \sqrt{1 - \frac{4}{M^2}}\right]}$	$-\frac{1}{2M^2 r} \cdot \frac{1}{r}$
f_n	$\frac{f_n}{f_s} = 1 + \frac{1}{2r}$	$\frac{1}{2M^2 r}\left(\frac{1}{r} + 1\right)$	$\frac{f_n}{f_s} = \sqrt{1 + \frac{1}{2r}\left[1 + \sqrt{1 + \frac{4}{M^2}}\right]}$	$\frac{1}{2M^2 r} \cdot \frac{1}{r}$
f_p	$\frac{f_p}{f_s} = 1 + \frac{1}{2r}$	$-\frac{1}{8r^2}$	$\frac{f_p}{f_s} = \sqrt{1 + \frac{1}{r}}$	0

Approximate Relations between the Characteristic Frequencies and the Series Resonance Frequency f_s of a Piezoelectric Vibrator

	$R_1 =$	$\frac{f^2_{mT}}{f^2_s} - 1 =$
COMPLETE SOLUTION	$\dfrac{R_T\,(2 - \surd)}{\frac{C_o^2}{C_T^2}\,\frac{b^2 \surd^2}{1 + M_T^2} - \frac{4b}{M_T^2}\,\frac{C_o}{C_T}\left(b\,\frac{C_o}{C_T} + 1\right) - \left(1 + \frac{1}{M_T^2}\right)}$ $\surd = \sqrt{\left[2 + \frac{R_{st}}{R_T}\left(1 + \frac{1}{M_T^2}\right)\right]^2 + \frac{4}{M_T^2}}$ (4)	$\dfrac{1}{rb} + \dfrac{\frac{b}{rM^2}\left(1 + C + \frac{2}{M_T^2}\right)}{B - 1 + \left(1 + \frac{1}{M_T^2}\right)\left(1 - \sqrt{\frac{1}{1 + \frac{1}{M_T^2}}\left[B^2 + \frac{4b^2}{M^2} C + \frac{1}{M_T^2}\left(1 + \frac{4b^2}{M^2}\right)\right]}\right)}$ $B = 1 + \frac{4b^2\,R_T}{M^2\,R_1};\ C = 1 + \frac{4b\,R_T}{MM_T\,R_1}$ (10)
$b = 0$	$\dfrac{R_T\left(\sqrt{\left[2 + \frac{R_{st}}{R_T}\left(1 + \frac{1}{M_T^2}\right)\right]^2 + \frac{4}{M_T^2}} - 2\right)}{1 + \frac{1}{M_T^2}}$ (4a)	$\dfrac{2\,R_T}{rMM_T\,R_1\left(1 + \frac{1}{M_T^2}\right)}$ (10a)
$C_T = 0$	$\dfrac{R_{st}}{1 - \frac{R_{st}^2}{X_o^2}\,b^2\left(4\,\frac{R_T}{R_{st}} + 1\right)}$ (4b)	$\dfrac{1}{rb} + \dfrac{2b}{rM^2 B\left(1 - \sqrt{1 + \frac{4b^2}{M^2\,B^2}}\right)}$ (10b)
$R_T = \infty$	$\dfrac{R_{st}\sqrt{1 + \left(\frac{2X_T}{R_{st}}\right)^2}}{1 + 4b\,\frac{C_o}{C_T} - \frac{R_{st}^2}{X_o^2}\,b^2}$ (4c)	$\dfrac{1}{rb} + \dfrac{2b\left(2b\,\frac{C_o}{C_T} + 1\right)}{rM^2\left(1 - \sqrt{\frac{4b^2}{M^2}\left(1 + 2b\frac{C_o}{C_T}\right)^2 + 1}\right)}$ (10c)
$\frac{4}{M_T^2} \ll 1$	$\dfrac{R_{st}}{1 + \frac{4b}{M_T^2}\,\frac{C_o}{C_T} - \frac{b^2}{M_T^2}\left(\frac{R_{st}\,C_o}{R_T\,C_T}\right)^2\left(1 + 4\,\frac{R_T}{R_{st}}\right)}$ (4d)	$\frac{2}{M^2 r}\,\frac{C_T\,R_T^2}{C_o\,R_1^2} - \frac{4b\,R_T}{M^2\,R_1 r} - \frac{b}{M^2 r}$ is valid for $b^2/M^2 \ll 1$ (10d)
$b = 0$; $R_T = \infty$	$R_{st}\sqrt{1 + \left(\frac{2X_T}{R_{st}}\right)^2}$ (4e)	$2\,\frac{C_1}{C_T}$ (10e)
$b = 0$; $C_T = 0$	R_{st} (4f)	0 (10f)

TABLE 5

Relationship Between Measured and Fundamental Values